

DIMENSION COUNTS FOR LIMIT LINEAR SERIES ON CURVES NOT OF COMPACT TYPE

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ABSTRACT. We first prove a generalized Brill-Noether theorem for linear series with prescribed multivanishing sequences on smooth curves. We then apply this theorem to prove that spaces of limit linear series have the expected dimension for a certain class of curves not of compact type, whenever the gluing conditions in the definition of limit linear series impose the maximal codimension. Finally, we investigate these gluing conditions in specific families of curves, showing expected dimension in several cases, each with different behavior. One of these families sheds new light on the work of Cools, Draisma, Payne and Robeva in tropical Brill-Noether theory.

1. INTRODUCTION

In [Oss14b], the author introduced a theory of limit linear series for nodal curves not of compact type, along with an equivalent definition, generalizing the Eisenbud-Harris definition, for curves of ‘pseudocompact type’, meaning that their dual graphs yield trees after collapsing all multiple edges. It was shown that linear series always specialize to limit linear series, and, as in the work of Eisenbud and Harris, for curves of pseudocompact type it was shown that limit linear series occurring in families of the expected dimension can be smoothed to linear series on smooth curves. In order to show that the theory is useful, it thus remains to show first that it is tractable, and second that – at least in some interesting cases – one does in fact obtain families of limit linear series having the expected dimension. This is precisely what is accomplished in the present paper.

We work in the context of curves of pseudocompact type, where a basic ingredient in the definition of limit linear series is the notion of ‘multivanishing sequence’, defined as follows:

Definition 1.1. Let X be a smooth projective curve, $r, d \geq 0$, and $D_0 \leq D_1 \leq \dots \leq D_{b+1}$ a sequence of effective divisors on X , with $D_0 = 0$ and $\deg D_{b+1} > d$. Given (\mathcal{L}, V) a \mathbf{g}_d^r on X , define the **multivanishing sequence** of (\mathcal{L}, V) along D_\bullet to be the sequence

$$a_0 \leq \dots \leq a_r$$

where a value a appears in the sequence m times if for some i we have $\deg D_i = a$, $\deg D_{i+1} > a$, and $\dim(V(-D_i)/V(-D_{i+1})) = m$.

In the above, if D is an effective divisor and (\mathcal{L}, V) a linear series, we write $V(-D)$ for $V \cap \Gamma(X, \mathcal{L}(-D))$.

Thus, this generalizes usual vanishing sequences, but also addresses a wide variety of geometric conditions, from secancy (in the sense of requiring two or more

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points to map to the same point) to bitangency, and so forth. The definition of limit linear series for curves of pseudocompact type (recalled in Definition 2.16 below) involves a condition on multivanishing sequences, as well as a gluing condition.

Our first result, given precisely in Theorem 3.3 below, is a generalized Brill-Noether theorem for linear series with imposed multivanishing sequences. We then apply this theorem to investigate which reducible nodal curves are **Brill-Noether general** – that is, have limit linear series spaces of the expected dimension. We prove in Theorem 4.1 that any curve of pseudocompact type whose components are Brill-Noether general with respect to imposed multivanishing is itself Brill-Noether general as long as the gluing conditions impose maximal codimension. These constitute the general results of the paper.

The remainder of the paper consists of studying two extremes where gluing becomes relatively tractable: when there are few nodes (at most two or three) between any given pair of components, or at the opposite extreme, ‘binary curves’ consisting of two rational components glued to one another at $g + 1$ nodes. In each situation, we are able to prove the necessary independence of gluing conditions under suitable hypotheses. Specifically, in order to define limit linear series on a given curve, we first choose two additional structures: a numerical “chain structure” which can be thought of as describing the singularities of a one-parameter smoothing of the curve, and an “enriched structure” consisting of a collection of twisting bundles which may likewise be obtained from a (regular) smoothing. Generality of enriched structure corresponds to a more typical algebrogeometric notion of generality. Imposing conditions on chain structures in some sense restricts to more special families of curves, but insofar as it can be used to restrict directions of approach to a given nodal curve, it can still be used to ensure that a space of limit linear series has the expected dimension. Indeed, the notion of generality in tropical Brill-Noether theory (as in Cools-Draisma-Payne-Robeva [CDPR12]) is essentially the same as restricting chain structures – see the end of §2.

In the case of few nodes, in Corollary 5.2 we produce families of curves which are Brill-Noether general when restrictions are placed on the chain structures, irrespective of enriched structure. This may be viewed as a generalization of [CDPR12], and in fact helps explain why their genericity condition should lead to good behavior. Then, in Corollary 6.2 we show that a certain narrower family of curves has the complementary behavior, with Brill-Noether generality occurring for general enriched structures, irrespective of the chain structure. Here, the family in question contains in particular the curves considered by Jensen and Payne in their work [JP] on a tropical approach to the Gieseker-Petri theorem. Finally, in Corollary 7.3 we show that for binary curves, the behavior is very different: whether or not the curve is Brill-Noether general is independent of both enriched structures and chain structures, and may be expressed in terms of usual linear series (in a certain restricted range of multidegrees) on the underlying curve. Using work of Caporaso [Cap10] we conclude that limit linear series spaces for $r \leq 2$ have the expected dimension on general binary curves.

This last result draws a distinction between our limit linear series and the notion introduced by Amini and Baker in [AB]: a general binary curve of genus at least 3 is not ‘hyperelliptic’ with respect to our definition of limit linear series, but it is with respect to the Amini-Baker definition. In fact, for us the main purpose of the final sections of the paper is to validate the definition of limit linear series given in

[Oss14b] by producing quite distinct infinite families of curves where the limit linear series spaces have the correct dimension. At the same time, as mentioned above, the particular families studied have also arisen (at least in special cases) elsewhere in the literature, leading to natural points of contact. Indeed, the relationship between Corollary 5.2 and the work in [CDPR12] is highly suggestive on the tropical side, pointing to families of graphs which are and are not likely to be Brill-Noether general in the tropical sense. In addition, we hope that our work will lead to a new proof that the tropical linear series studied in [CDPR12] can all be smoothed, as was recently proved by Cartwright, Jensen and Payne in [CJP]. We will investigate this and other aspects of the relationship to the tropical theory in [Oss14a]. Finally, our work leads to new explicit criteria for Brill-Noether generality in terms of special fibers of degenerations, stated in Corollary 5.5.

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Conventions. We do not assume base fields to be algebraically closed, but we do assume that our nodal curves are split over their base field, meaning that all components and nodes are defined over the base field.

If v is the vertex of the dual graph of a nodal curve, we let Z_v denote the corresponding component of the curve. If e is an edge, we let P_e denote the corresponding node.

If e is an edge of a directed graph, we denote by $h(e)$ and $t(e)$ the head and tail of e respectively.

2. BACKGROUND ON LIMIT LINEAR SERIES

In this section we recall the definition of limit linear series introduced in [Oss14b], simplifying somewhat as a consequence of restricting to the context of curves of pseudocompact type. Although our presentation is mathematically self contained, we refer the reader to §2 of [Oss14b] for additional remarks and examples. However, for the benefit of readers coming from tropical geometry we do include a rough dictionary between our definitions and those arising in tropical Brill-Noether theory at the end of this section.

We begin with some definitions of a combinatorial nature. In the below, Γ will be obtained by choosing a directed structure on the dual graph of a curve of pseudocompact type. The following definition forms the basis for our approach to keeping track of chains of rational curves inserted at the nodes of the original curve.

Definition 2.1. A **chain structure** on Γ is a function $\mathbf{n} : E(\Gamma) \rightarrow \mathbb{Z}_{>0}$.

The chain structure will determine the length of the chain of rational curves inserted at a given node; for notational convenience, the trivial case (in which no rational curves are inserted) corresponds to $\mathbf{n}(e) = 1$.

We work in the following situation throughout.

Situation 2.2. Let Γ be a directed graph without loops, and \mathbf{n} a chain structure on Γ . For each pair of an edge e and adjacent vertex v of Γ , let $\sigma(e, v) = 1$ if e has tail v , and -1 if e has head v .

Let $\bar{\Gamma}$ be the graph obtained from Γ by collapsing all multiple edges, and assume that $\bar{\Gamma}$ is a tree.

Definition 2.3. An **admissible multidegree** w of total degree d on (Γ, \mathbf{n}) consists of a function $w_\Gamma : V(\Gamma) \rightarrow \mathbb{Z}$ together with a tuple $(\mu(e))_{e \in E(\Gamma)}$, where each $\mu(e) \in \mathbb{Z}/\mathbf{n}(e)\mathbb{Z}$, such that

$$d = \#\{e \in E(\Gamma) : \mu(e) \neq 0\} + \sum_{v \in V(\Gamma)} w_\Gamma(v).$$

The idea behind admissible multidegrees is that in order to extend line bundles, we need only consider multidegrees which have degree 0 or 1 on each rational curve inserted at the node, with degree 1 occurring at most once in each chain. Thus, $\mu(e)$ determines where on the chain (if anywhere) positive degree occurs. See Definition 2.11 below for details.

We now define twists of multidegrees.

Definition 2.4. If (e, v) is a pair of an edge e and an adjacent vertex v of $\bar{\Gamma}$, given an admissible multidegree w , we define the **twist** of w at (e, v) to be obtained from w as follows: for each \tilde{e} of Γ over e , increase $\mu(\tilde{e})$ by $\sigma(\tilde{e}, v)$. Now, decrease $w_\Gamma(v)$ by the number of \tilde{e} for which $\mu(\tilde{e})$ had been equal to 0, and for each \tilde{e} , if the new $\mu(\tilde{e})$ is zero, increase $w_\Gamma(v')$ by 1, where v' is the other vertex adjacent to v .

Twists will be the change in multidegrees accomplished by twisting by certain natural line bundles; see Notation 2.12 below.

Definition 2.5. An admissible multidegree w is **concentrated** at a vertex $v \in V(\Gamma)$ if for each $v' \neq v$, we have that w is negative in index v' after twisting at (e, v') , where e is the edge of $\bar{\Gamma}$ from v' in the direction of v .

For the sake of simplicity, the above definition is slightly more restrictive than that of [Oss14b]; see Remark 2.18. It generalizes the situation for Eisenbud and Harris of considering line bundles with degree d on one component and degree 0 on the others. If an admissible multidegree w_v is concentrated at v and also has negative degree in index v , then it will lead to a vacuous theory of limit linear series, because of Proposition 3.3 of [Oss14b]. Accordingly, we will always assume implicitly that any such w_v is nonnegative in index v .

We will work throughout in the following situation.

Situation 2.6. Suppose we are given an admissible multidegree w_0 , and let $(w_v)_{v \in V(\Gamma)} = (w_v, \mu_v(\bullet))_v$ be a collection of admissible multidegrees, each obtained by w_0 by successive twists, and such that:

- (I) each w_v is concentrated at v ;
- (II) for each $v, v' \in V(\bar{\Gamma})$ connected by an edge e , the multidegree $w_{v'}$ is obtained from w_v by twisting $b_{v,v'}$ times at (e, v) , for some $b_{v,v'} \in \mathbb{Z}_{\geq 0}$.

The below graph will help us keep track of the relevant multidegrees.

Definition 2.7. In Situation 2.6, let

$$V(\bar{G}(w_0)) \subseteq \mathbb{Z}^{V(\Gamma)} \times \prod_{e \in E(\Gamma)} \mathbb{Z}/\mathbf{n}(e)\mathbb{Z}$$

consist of admissible multidegrees w such that there exist $v, v' \in V(\bar{\Gamma})$ connected by some edge e , with w obtainable from w_v by twisting b times at (e, v) , for some b with $0 \leq b \leq b_{v,v'}$.

There is an edge ϵ from w to w' in $\bar{G}(w_0)$ if there exists (e, v) in $\bar{\Gamma}$ such that w' is obtained from w by twisting at (e, v) .

Thus, $\tilde{G}(w_0)$ is essentially a tree, obtained from $\bar{\Gamma}$ by subdividing each edge into $b_{v,v'}$ edges, and replacing each resulting edge with a pair of edges going in opposite directions.

We now move on to the geometric constructions which underlie our definition of limit linear series. First, in the non-compact-type case, additional structure beyond the underlying nodal curve is necessary in order to make a useful definition.

Definition 2.8. If X' is a projective nodal curve with dual graph Γ' , an **enriched structure** on X' consists of the data, for each $v \in V(\Gamma')$ of a line bundle \mathcal{O}_v on X' , satisfying the following conditions:

(I) for any $v \in V(\Gamma')$, we have

$$\mathcal{O}_v|_{Z_v} \cong \mathcal{O}_{Z_v}(-(Z_v^c \cap Z_v)), \text{ and } \mathcal{O}_v|_{Z_v^c} \cong \mathcal{O}_{Z_v^c}(Z_v^c \cap Z_v);$$

(II) we have

$$\bigotimes_{v \in V(\Gamma')} \mathcal{O}_v \cong \mathcal{O}_X.$$

In the above, Z_v^c is the closure of the complement of Z_v .

The curve on which we place an enriched structure will not be the original nodal curve, but the following curve obtained by also taking the chain structure into account.

Definition 2.9. Given a projective nodal curve X with dual graph Γ and a chain structure \mathbf{n} , let \tilde{X} denote the nodal curve obtained from X by, for each $e \in E(\Gamma)$, inserting a chain of $\mathbf{n}(e) - 1$ projective lines at the corresponding node. Let $\tilde{\Gamma}$ be the dual graph of \tilde{X} , with a natural inclusion $V(\Gamma) \subseteq V(\tilde{\Gamma})$.

Finally, our definition of limit linear series will occur in the following context.

Situation 2.10. In Situation 2.6, suppose further that we have a projective nodal curve X over a field k with dual graph Γ , and an enriched structure $(\mathcal{O}_v)_v$ on the corresponding \tilde{X} . Fix also $r > 0$, and let d be the total multidegree of w_0 .

We now describe how our combinatorial notions of multidegrees and twists arise in the geometric setting.

Definition 2.11. Using our orientation of $E(\Gamma)$, an admissible multidegree w of total degree d on (X, \mathbf{n}) gives a multidegree of total degree d on \tilde{X} by assigning, for each $e \in E(\Gamma)$, degree 0 on each component of the corresponding chain of projective curves, except for degree 1 on the $\mu(e)$ th component when $\mu(e) \neq 0$.

The following notation will not be used later, and is necessary only to set up Notation 2.12. In Situation 2.10, for any edge $\varepsilon \in E(\tilde{G}(w_0))$, starting at $w = (w_\Gamma, (\mu(e))_{e \in E(\Gamma)})$ and determined by twisting at (e, v) , let $\tilde{\Gamma}'$ be the graph obtained from $\tilde{\Gamma}$ by removing, for each edge \tilde{e} of Γ lying over e , the $(\sigma(\tilde{e}, v)\mu(\tilde{e}) + 1)$ st edge of $\tilde{\Gamma}$ lying over \tilde{e} , starting from v . Then let $S \subseteq V(\tilde{\Gamma})$ consist of the vertices in the connected component of $\tilde{\Gamma}'$ containing v . Next, let \mathcal{O}_ε be the twisting line bundle on \tilde{X} defined by

$$\mathcal{O}_\varepsilon := \bigotimes_{v' \in S} \mathcal{O}_{v'}.$$

Similarly, given $w, w' \in V(\bar{G}(w_0))$, let $P = (\varepsilon_1, \dots, \varepsilon_m)$ be a minimal path from w to w' in $\bar{G}(w_0)$, and set

$$\mathcal{O}_{w,w'} = \bigotimes_{i=1}^m \mathcal{O}_{\varepsilon_i}.$$

The point of this construction is that twisting a line bundle by \mathcal{O}_ε changes its multidegree in the same manner as twisting by (e, v) , so that if \mathcal{L} has multidegree w , then $\mathcal{L} \otimes \mathcal{O}_{w,w'}$ has multidegree w' .

Notation 2.12. In Situation 2.10, suppose \mathcal{L} is a line bundle on \tilde{X} of multidegree w_0 . Then for any $w \in V(\bar{G}(w_0))$, set

$$\mathcal{L}_w := \mathcal{L} \otimes \mathcal{O}_{w_0,w}.$$

For $v \in V(\Gamma)$, set

$$\mathcal{L}^v := \mathcal{L}_{w_v}|_{Z_v}.$$

The following divisor sequences derived from our combinatorial data will provide the backdrop to the multivanishing sequences and gluing conditions considered in our definition of limit linear series.

Notation 2.13. In Situation 2.10, for each pair (e, v) of an edge and adjacent vertex of $\bar{\Gamma}$, let $D_0^{(e,v)}, \dots, D_{b_{v,v'}+1}^{(e,v)}$ be the sequence of effective divisors on Z_v defined by $D_0^{(e,v)} = 0$, and for $i \geq 0$,

$$D_{i+1}^{(e,v)} - D_i^{(e,v)} = \sum_{\substack{\tilde{e} \text{ over } e : \\ \sigma(\tilde{e}, v)\mu_v(\tilde{e}) \equiv -i \pmod{\mathbf{n}(\tilde{e})}}} P_{\tilde{e}},$$

where $P_{\tilde{e}}$ denotes the node of X corresponding to \tilde{e} .

A global line bundle, together with our twisting bundles, induce gluing isomorphisms as follows.

Proposition 2.14. *Given \mathcal{L} on \tilde{X} of multidegree w_0 , and vertices v, v' of $\bar{\Gamma}$ connected by an edge e , then for $i = 0, \dots, b_{v,v'}$ we have isomorphisms*

$$\varphi_i^{(e,v)} : \mathcal{L}^v(-D_i^{(e,v)})/\mathcal{L}^v(-D_{i+1}^{(e,v)}) \xrightarrow{\sim} \mathcal{L}^{v'}(-D_{b_{v,v'}-i}^{(e,v')})/\mathcal{L}^{v'}(-D_{b_{v,v'}+1-i}^{(e,v')})$$

induced by the line bundle $\mathcal{L}_{w(v,v',i)}$, where $w(v, v', i) \in V(\bar{G}(w_0))$ is the i th vertex between w_v and $w_{v'}$.

This is essentially the last part of Proposition 4.4 of [Oss14b].

Definition 2.15. If D_0, \dots, D_{b+1} is a non-decreasing sequence of effective divisors on a smooth proper curve, we say j is **critical** for D_\bullet if $D_{j+1} \neq D_j$.

We can now give our definition of limit linear series.

Definition 2.16. In Situation 2.10, suppose we have a tuple $(\mathcal{L}, (V^v)_{v \in V(\Gamma)})$ with \mathcal{L} a line bundle of multidegree w_0 on \tilde{X} , and each V^v an $(r+1)$ -dimensional space of global sections of the resulting \mathcal{L}^v . For each pair (e, v) in $\bar{\Gamma}$, let $a_0^{(e,v)}, \dots, a_r^{(e,v)}$ be the multivanishing sequence of V^v along $D_\bullet^{(e,v)}$. Then $(\mathcal{L}, (V^v)_{v \in V(\Gamma)})$ is a **limit linear series** if for any $e \in E(\Gamma)$, with adjacent vertices v, v' , we have:

- (I) for $\ell = 0, \dots, r$, if $a_\ell^{(e,v)} = \deg D_j^{(e,v)}$ with j critical for $D_\bullet^{(e,v)}$, then
- $$(2.1) \quad a_{r-\ell}^{(e,v')} \geq \deg D_{b_{v,v'}-j}^{(e,v')};$$
- (II) there exist bases $s_0^{(e,v)}, \dots, s_r^{(e,v)}$ of V^v and $s_0^{(e,v')}, \dots, s_r^{(e,v')}$ of $V^{v'}$ such that
- $$\text{ord}_{D_\bullet^{(e,v)}} s_\ell^{(e,v)} = a_\ell^{(e,v)}, \quad \text{for } \ell = 0, \dots, r,$$
- and similarly for $s_\ell^{(e,v')}$, and for all ℓ with (2.1) an equality, we have
- $$\varphi_j^{(e,v)}(s_\ell^{(e,v)}) = s_{r-\ell}^{(e,v')}$$
- when we consider $s_\ell^{(e,v)} \in V^v(-D_j^{(e,v)})$ and $s_{r-\ell}^{(e,v')} \in V^{v'}(-D_{b_{v,v'}-j}^{(e,v')})$, where j is as in (I), and $\varphi_j^{(e,v)}$ is as in Proposition 2.14.

We say a limit linear series is **refined** if (2.1) holds with equality for all ℓ .

Notation 2.17. In Situation 2.10, let

$$G_{\bar{w}_0}^r(X, \mathbf{n}, (\mathcal{O}_v)_v)$$

denote the moduli scheme of limit linear series.

The notation \bar{w}_0 reflects that the space $G_{\bar{w}_0}^r(X, \mathbf{n}, (\mathcal{O}_v)_v)$ depends on w_0 only up to arbitrary twists. For the construction of this moduli scheme, see §3 of [Oss14b].

Remark 2.18. Although our definition of concentrated is more restrictive than that of [Oss14b], this does not cause any technical difficulties. Indeed, it is easy to see that tuples $(w_v)_v$ as in Situation 2.6 always exist despite our more restrictive definition, and Proposition 3.5 of [Oss14b] asserts that the notion of limit linear series is in fact independent of the choice of tuple $(w_v)_v$. However, we briefly explain the relationship between our present definition and the definition in [Oss14b].

If we wanted an equivalent definition of concentrated to that given in [Oss14b], we could say that w is concentrated at a vertex $v \in V(\Gamma)$ if there is an ordering

$$V(\Gamma) = \{v_1, v_2, \dots\}$$

with $v = v_1$, and such that for each $i > 1$, we have that w becomes negative in index v_i after taking the composition of the twists at (e_j, v_i) over all $j < i$ with v_j adjacent to v_i , where e_j is the edge of $\bar{\Gamma}$ connecting v_i to v_j . Indeed, this is equivalent to the definition in [Oss14b] because the degree in index v_i is the same after the above-described twists as after taking the negative twists at all v_j for $j < i$. We thus see that the definition we are using is more restrictive, as claimed, since it is equivalent to considering an ordering which is consistent with the distance from v in $\bar{\Gamma}$.

Relationship to tropical Brill-Noether theory. Although our theory of limit linear series is quite different from the theory of divisors on graphs, certain definitions translate directly, as we now explain.

First, our chain structures correspond to lengths of edges of metric graphs, with the distinction that we restrict to integer lengths. More specifically, if we have a one-parameter smoothing of our nodal curve X , then both the chain structure and the associated metric graph are determined by looking at the number of exceptional components lying over each node of X in a resolution to a regular smoothing.

Next, our notion of admissible multidegree corresponds to divisors on the graph $\tilde{\Gamma}$, with the added constraint that the divisor must be nonnegative on all vertices of $\tilde{\Gamma}$ lying over edges of Γ , and the vertices over a given edge can have total degree at most 1.

Our notion of an admissible multidegree being concentrated at v then corresponds roughly to a v -reduced divisor on $\tilde{\Gamma}$ – a v -reduced divisor is concentrated at v , but not necessarily conversely (for instance, we do not require nonnegativity).

Finally, our twists of multidegrees at (e, v) correspond to chip-firings along the set of vertices on the same side of e as v is. Thus, our $\tilde{G}(w_0)$ consists of a subset of the divisors on $\tilde{\Gamma}$ linearly equivalent to w_0 .

3. LINEAR SERIES WITH IMPOSED MULTIVANISHING

In this section, we prove a generalized Brill-Noether theorem for linear series with prescribed multivanishing, generalizing the theorem of Eisenbud and Harris (Theorem 4.5 of [EH86]) from the case of usual vanishing sequences to the case of multivanishing sequences. We prove the genus-0 case with at most two multivanishing sequences directly, and then use limit linear series for curves of compact type to prove the main theorem.

We begin by setting up some basic notation and definitions.

Notation 3.1. Let X be a smooth projective curve of genus g , and fix integers $r, d, n > 0$, and for $i = 1, \dots, n$ fix also nondecreasing sequences D_\bullet^i of effective divisors on X , such that the support of D_\bullet^i is disjoint from that of $D_\bullet^{i'}$ for every $i \neq i'$. Fix also a tuple of nondecreasing sequences a^i , such that for each i, j we have $a_j^i = \deg D_\ell^i$ for some ℓ critical for D_\bullet^i , and the number of repetitions of a_j^i is at most $\deg D_{\ell+1}^i - D_\ell^i$.

Then we denote by $G_d^r(X, (D_\bullet^i, a^i)_i)$ the space of \mathfrak{g}_d^r s on X having multivanishing sequence at least a^i along D_\bullet^i for each i .

Definition 3.2. Let X be a smooth projective curve of genus g , and fix integers $r, d, n > 0$, and for $i = 1, \dots, n$ fix also $m_i > 0$. Choose distinct points P_j^i on X for $i = 1, \dots, n$ and $j = 1, \dots, m_i$. Then we say that $(X, (P_j^i)_{i,j})$ is **strongly Brill-Noether general** for r, d if, for all tuples of nondecreasing effective divisor sequences D_\bullet^i , such that every divisor in D_\bullet^i is supported among $P_1^i, \dots, P_{m_i}^i$, and for every tuple of nondecreasing sequences a^i as in Notation 3.1, the space $G_d^r(X, (D_\bullet^i, a^i)_i)$ has the expected dimension

$$\rho := g + (r+1)(d-r-g) - \sum_{i=1}^n \left(\sum_{j=0}^r (a_j^i - j) + \sum_{\ell=0}^{b_i} \binom{r_\ell^i}{2} \right)$$

if it is nonempty.

In the above, D_\bullet^i is indexed from 0 to $b_i + 1$, and r_ℓ^i is defined to be 0 if ℓ is not critical for D_\bullet^i , and the number of times $\deg D_\ell^i$ occurs in a^i if ℓ is critical.

We will sometimes refer to an X together with choices of P_j^i as above as a ‘multimarked curve.’

The following is then the main theorem of this section. We also take the opportunity to state some specific situations in which Brill-Noether generality is known under precisely stated conditions.

Theorem 3.3. *With notation as in Definition 3.2, we have that $(X, (P_j^i)_{i,j})$ is strongly Brill-Noether general for r and d if any of the following conditions are satisfied:*

- (I) *either $\text{char } k = 0$ or $\text{char } k = p > d$, and both X and the P_j^i are general;*
- (II) *$n \leq 2$ and both X and the P_j^i are general;*
- (III) *$g = 0$ and $n \leq 2$;*
- (IV) *either $\text{char } k = 0$ or $\text{char } k = p > d$, $g = 0$, and all m_i are equal to 1;*
- (V) *$g = 1$, $n \leq 2$, $m_i = 1$ for each i , and if $n = 2$, then $P_1^1 - P_2^1$ is not ℓ -torsion for any $\ell \leq d$;*
- (VI) *$g = 2$, $n \leq 1$, and if $n = 1$, then $m_1 = 1$ and P_1 is not a Weierstrass point.*

Cases (I) and (II) should be considered the main result of the theorem. Cases (IV)-(VI) fall into the case of classical vanishing sequences, and thus were already known. The reason we restate them here is that they will substantially broaden the number of explicit reducible curves for which we can show that limit linear series spaces have the expected dimension in Corollary 5.5 below. We also mention that the $g = 0$ case of (I) was already proved by García-Puente *et. al.* in [GPHH⁺12].

Note also that our characteristic hypotheses are essentially optimal, as even in the special case treated by Eisenbud and Harris, the characteristic hypotheses are already necessary, even in genus 0 for $r = 1$.

Proof. We first consider case (III). Of course, if $n = 0$ there is nothing to show, and if $n = 1$ the imposition of the multivanishing sequence a^1 is just a Schubert cycle in $G(r+1, \Gamma(X, \mathcal{O}(d)))$, of codimension $\sum_{j=0}^r (a_j^1 - j) + \sum_{\ell=0}^{b_1} \binom{r_\ell^1}{2}$. Indeed, such a Schubert cycle for a partial flag is the same as the Schubert cycle obtained from any completion of the flag, where any repetitions $a_{j_1} = a_{j_1+1} = \dots = a_{j_2}$ are replaced by $a_{j_1}, a_{j_1} + 1, \dots, a_{j_1} + j_2 - j_1$. Similarly, in the case $n = 2$, because the support of D_\bullet^1 is disjoint from that of D_\bullet^2 , we see that the corresponding flags meet transversely, so the associated Schubert cycles intersect in the expected dimension, as desired.

We now prove cases (I) and (II). If X is general, then for n general marked points Q_1, \dots, Q_n , under our hypotheses we have that the space of linear series on X with prescribed ramification at the Q_i has the expected dimension; see for instance [Oss] (but note that the proof of the main theorem can be simplified now that it is possible to construct a proper moduli space of limit linear series in smoothing families, as carried out in [Oss14c]). Accordingly, consider a one-parameter family in which for each i , all the points in the support of D_\bullet^i approach Q_i . Blowing up the special fiber at the Q_i , we obtain a curve X_0 consisting of a copy of X , together with rational tails glued at each Q_i , and with each D_\bullet^i specializing onto the corresponding rational tail. But by construction and the previously addressed rational case, we see that the space of limit linear series on X_0 with imposed multivanishing at the (limits of the) D_\bullet^i has the desired dimension, and the theorem follows.

As mentioned above, cases (IV)-(VI) were previously known: see for instance Theorem 2.3 of [EH83], Lemma 2.1 of [Oss], and Theorem 1.1 of [EH87]. Thus, the theorem is proved. \square

4. EXPECTED DIMENSION FOR LIMIT LINEAR SERIES

In a sense, the smoothing theorem (Theorem 6.1) of [Oss14b] already says that the “expected dimension” of the space of limit linear series on a curve of pseudo-compact type is ρ , in the sense that it is at least ρ , and if the conditions cutting it out have maximal codimension, then it is exactly ρ . However, these conditions – “linked determinantal loci” – are rather abstract, so it is the goal of the present section to use the alternative definition of limit linear series to give more geometrically concrete criteria for when the dimension is as expected. We then apply these criteria in subsequent sections to more explicit families of curves. Our main result is as follows:

Theorem 4.1. *In Situation 2.10, suppose further that each component of X (considered as a multimarked curve) is strongly Brill-Noether general for r and d . Then the limit linear series space $G_{\bar{w}_0}^r(X, \mathbf{n}, (\mathcal{O}_v)_v)$ has dimension ρ if the gluing conditions imposed by Definition 2.16 (II) impose the maximal possible codimension. Furthermore, in this case the refined limit linear series are dense.*

Recall that a number of sufficient conditions for strong Brill-Noether generality are listed in Theorem 3.3 above.

The main ingredient in Theorem 4.1 is a combinatorial calculation which is the analogue of the Eisenbud-Harris “additivity of the Brill-Noether number” (Proposition 4.6 of [EH86]). Despite the restriction to the pseudocompact-type case, the calculation is considerably more complicated, due to the presence of repeated vanishing orders and gluing conditions.

Lemma 4.2. *In the situation of Notation 2.13, let Z_1 and Z_2 be components of X corresponding to vertices v, v' of $\bar{\Gamma}$ connected by an edge e . Write $D_\bullet^1 := D_\bullet^{(e,v)}$ and $D_\bullet^2 := D_\bullet^{(e,v')}$, let $\{0, \dots, b+1\}$ be the index set for D_\bullet^1 , and let C be the subset of critical indices. Observe that $\deg D_j^1 + \deg D_{b+1-j}^2$ is independent of j , and denote its common value by c . For each j , also set*

$$f_j := \deg D_{c+1-j}^2 - \deg D_{c-j}^2 = \deg D_{j+1}^1 - \deg D_j^1.$$

Given also sequences a^1, a^2 satisfying the conditions of Notation 3.1, let r_j^i be as in Definition 3.2. Finally, set

$$g_j = r_j^1 + \sum_{m \in C, m < j} (r_m^1 - r_{b-m}^2).$$

We then have

(4.1)

$$\sum_{j \in C} g_j (f_j + g_j - r_j^1 - r_{b-j}^2) \geq (r+1)(c-1) - \sum_{\ell=0}^r (a_\ell^1 + a_{r-\ell}^2) - \sum_{j \in C} \left(\binom{r_j^1}{2} + \binom{r_j^2}{2} \right),$$

with equality precisely in the refined case.

One checks using (4.4) from the proof of Lemma 4.6 of [Oss14b] that the g_j defined above are precisely the number of sections s_ℓ for which the gluing condition of Definition 2.16 (II) is imposed in index j .

Proof. We first manipulate the righthand side of (4.1). We have

$$\begin{aligned}
 (r+1)(c-1) - \sum_{\ell=0}^r (a_\ell^1 + a_{r-\ell}^2) - \sum_{j \in C} \left(\binom{r_j^1}{2} + \binom{r_j^2}{2} \right) \\
 = (r+1)(c-1) - \sum_{j \in C} (r_j^1 \deg D_j^1 + r_{b-j}^2 \deg D_{b-j}^2) - \sum_{j \in C} \left(\binom{r_j^1}{2} + \binom{r_j^2}{2} \right) \\
 (4.2) \quad = (r+1)(c-1) - \sum_{j \in C} (r_j^1 (c - f_j - \deg D_{b-j}^2) + r_{b-j}^2 \deg D_{b-j}^2) \\
 \quad + \sum_{j \in C} \left(\frac{(r_j^1)^2}{2} - \frac{r_j^1}{2} + \frac{(r_{b-j}^2)^2}{2} - \frac{r_{b-j}^2}{2} \right) \\
 = \sum_{j \in C} \left(f_j r_j^1 + \deg D_{b-j}^2 (r_j^1 - r_{b-j}^2) - \frac{(r_j^1)^2}{2} - \frac{(r_{b-j}^2)^2}{2} \right).
 \end{aligned}$$

First, in the refined case, we have $r_j^1 = r_{b-j}^2 = g_j$ for all $j \in C$, so we see from (4.2) that both sides of (4.1) are equal to

$$\sum_{j \in C} (r_j^1 f_j - (r_j^1)^2),$$

and in fact (4.1) is an identity. We next reduce to the refined case.

In the general case, we claim that if the sequences a^1, a^2 are not refined, then we can always decrease one of the a_ℓ^2 and still have an allowable sequence. Indeed, if ℓ_0 is maximal such that (2.1) is strict, then suppose that $a_{\ell_0}^1 = \deg D_{j_0}^1$, and $a_{r-\ell_0}^2 = \deg D_{b-j_1}^2$, with $j_1, j_0 \in C$, and $j_1 < j_0$ by hypothesis. Let $j_2 \in C$ be minimal greater than j_1 . Then our specific claim is that if we set $\hat{a}_{r-\ell}^2$ to be equal to $a_{r-\ell}^2$ for $\ell \neq \ell_0$, and $\hat{a}_{r-\ell_0}^2 = \deg D_{b-j_2}^2$, then we still have a valid sequence. Certainly, (2.1) will still be satisfied, and \hat{a}^2 is nondecreasing, so it is enough to see that $\hat{a}_{r-\ell_0}^2$ does not contain too many repetitions of $\deg D_{b-j_2}^2$. Now, the maximal number of allowed repetitions of $\deg D_{b-j_2}^2$ is

$$\deg D_{b+1-j_2}^2 - \deg D_{b-j_2}^2 = \deg D_{j_2+1}^1 - \deg D_{j_2}^1 = f_{j_2}.$$

Certainly, if $a_{r-\ell}^2 = \deg D_{b-j_2}^2$, then $\ell > \ell_0$; by our hypothesis for the maximality of ℓ_0 , for all such ℓ we have $a_\ell^1 = \deg D_{j_2}^1$. Now, if $j_0 > j_2$, we see that in fact there is no such ℓ , because a^1 is nondecreasing. On the other hand, if $j_0 = j_2$, then we must have $a_{\ell_0+f_{j_0}}^1 > \deg D_{j_0}^1$, so by the maximality of ℓ_0 , we have $a_{r-\ell_0-f_{j_0}}^2 < \deg D_{b-j_0}^2 = \deg D_{b-j_2}^2$, so we have at most $f_{j_0} - 1 = f_{j_2} - 1$ repetitions of $\deg D_{b-j_2}^2$ in a^2 , as desired.

Now, we examine the effect of replacing a^2 by \hat{a}^2 on both sides of (4.1). Clearly, the r_j^1 are unaffected, while we are decreasing $r_{b-j_1}^2$ by 1, and increasing $r_{b-j_2}^2$ by 1. Recalling that j_1 and j_2 are consecutive in C , we see that under our modification, g_{j_2} increases by 1, and the other g_j are unchanged. From this, we compute that the lefthand side of (4.1) increases by $g_{j_1} + g_{j_2} + f_{j_2} - r_{j_2}^1 - r_{b-j_2}^2$, while using (4.2), the righthand side increases by

$$\deg D_{b-j_1}^2 - \deg D_{b-j_2}^2 + r_{b-j_1}^2 - r_{b-j_2}^2 - 1 = f_{j_2} + r_{b-j_1}^2 - r_{b-j_2}^2 - 1.$$

Now, we have $g_{j_1} \leq r_{b-j_1}^2$ and $g_{j_2} \leq r_{j_2}^1$, but in fact both inequalities are strict, by considering the corresponding inequalities in our modified sequence. We thus see that under our modification, the righthand side of (4.1) increases by more than the lefthand side. Now, after finitely many iterations, we will reach the refined case, where we know both sides are equal, so we conclude that in general, the righthand side is at most equal to the lefthand side, with equality precisely in the refined case, as desired. \square

Proof of Theorem 4.1. By the definition of strong Brill-Noether generality, it is enough to verify that combinatorially, the expected dimension of $G_{w_\bullet}^r(X, \mathbf{n}, (\mathcal{O}_v)_v)$ is bounded by ρ , with strict inequality for non-refined choices of multivanishing sequences. For $v \in V(\Gamma)$, let d_v be the value of w_v in index v , and g_v the genus of Z_v . We stratify $G_{w_\bullet}^r(X, \mathbf{n}, (\mathcal{O}_v)_v)$ by multivanishing sequences, so fix sequences $a^{(e,v)}$ for each adjacent pair (e, v) in $\bar{\Gamma}$, satisfying (2.1). Then $G_{w_\bullet}^r(X, \mathbf{n}, (\mathcal{O}_v)_v)$ has data consisting of a tuple of $\mathfrak{g}_{d_v}^r$ s on each Z_v and gluing data for the underlying line bundle, and imposed conditions are given by (2.1) together with the gluing condition described in Definition 2.16 (II).

The expected dimension of the data is

$$\begin{aligned} \sum_v (g_v + (r+1)(d_v - r - g_v)) + n &= g + (r+1) \left(\sum_v d_v - r|V(\Gamma)| - g + n \right) \\ (4.3) \quad &= g + (r+1) \left(\sum_v d_v + |E(\Gamma)| - (r+1)|V(\Gamma)| - g + 1 \right) \end{aligned}$$

where $n = |E(\Gamma)| - |V(\Gamma)| + 1$ is the number of parameters obtained in gluing the line bundles on the Z_v to obtain a line bundle on X , and hence $n + \sum_v g_v = g$.

Now, we need to relate the d_v 's to d . Fix a v . By definition, w_v has entry d_v in index v . We also see that for any $v' \neq v$, if v'' is the vertex adjacent to v' in the direction of v , and $n_{v',v''}$ denotes the number of edges of Γ connecting v' to v'' , then the entry of w_v in index v' plus the number of edges e of Γ connecting v' to v'' such that $\mu_v(e) \neq 0$ is equal to

$$d_{v'} - c_{v',v''} + n_{v',v''},$$

where $c_{v',v''}$ is obtained as the c of Lemma 4.2 when we consider $Z_{v'}$ and $Z_{v''}$. By definition, if we sum over all v' , we get d , so we conclude that

$$(4.4) \quad d = \sum_{v \in V(\Gamma)} d_v - \sum_{e \in E(\bar{\Gamma})} c_{h(e),t(e)} + |E(\Gamma)|.$$

Thus, we can re-express our earlier expected dimension for the limit linear series data as

$$\begin{aligned} g + (r+1) \left(d + \sum_{e \in E(\bar{\Gamma})} c_{h(e),t(e)} - (r+1)|V(\Gamma)| - g + 1 \right) \\ = \rho + (r+1) \sum_{e \in E(\bar{\Gamma})} (c_{h(e),t(e)} - (r+1)). \end{aligned}$$

We next consider the codimension imposed by the multivanishing sequences. For each pair v, v' of adjacent vertices connected by an edge e of $\bar{\Gamma}$, the imposed

multivanishing on Z_v and $Z_{v'}$ at the nodes corresponding to e has codimension

$$\begin{aligned} & \sum_{\ell=0}^r (a_\ell^{(e,v)} - \ell + a_{r-\ell}^{(e,v')} - (r - \ell)) + \sum_{j=0}^{b_{v,v'}} \left(\binom{r_j^{(e,v)}}{2} + \binom{r_j^{(e,v')}}{2} \right) \\ &= -r(r+1) + \sum_{\ell=0}^r (a_\ell^{(e,v)} + a_{r-\ell}^{(e,v')}) + \sum_{j=0}^{b_{v,v'}} \left(\binom{r_j^{(e,v)}}{2} + \binom{r_j^{(e,v')}}{2} \right). \end{aligned}$$

Subtracting this from our previous expected dimension, we find that what remains is

$$\rho + \sum_{e \in E(\bar{\Gamma})} \left((r+1)(c_{h(e),t(e)} - 1) - \sum_{\ell=0}^r (a_\ell^{(e,v)} + a_{r-\ell}^{(e,v')}) - \sum_{j=0}^{b_{v,v'}} \left(\binom{r_j^{(e,v)}}{2} + \binom{r_j^{(e,v')}}{2} \right) \right).$$

But note that this is greater than ρ by precisely the sum over $e \in E(\bar{\Gamma})$ of the righthand side of (4.1). In order to confirm that we have the correct expected dimension, it therefore suffices to confirm that the expected codimension of the gluing conditions is at least the lefthand side of (4.1). In fact, we show that they are the same, with the exception that any negative summands in the lefthand side must be replaced by 0.

But this is clear: each gluing condition is of the form that two subspaces, of dimensions r_j^1 and r_{b-j}^2 respectively, of an f_j -dimensional vector space, must intersect in dimension at least g_j . That is, it is the preimage of a closed subset $Z \subseteq G(r_j^1, f_j) \times G(r_{b-j}^2, f_j)$ which one easily verifies is irreducible of codimension $g_j(f_j + g_j - r_j^1 - r_{b-j}^2)$ when the latter is nonnegative, yielding the desired statement. \square

5. CURVES WITH FEW NODES: RESTRICTED CHAIN STRUCTURES

We now consider curves which are close to being of compact type, in the sense that any two components intersect in at most three nodes. For such curves, individual gluing conditions are straightforward to understand, so in light of Theorem 4.1, if we also impose strong Brill-Noether generality on the individual components, in order to prove the entire curve is Brill-Noether general, it is enough to show that the gluing conditions are independent of one another. In this section, we show that suitable conditions on the chain structures always imply the desired independence, irrespective of enriched structures. The families we consider generalize the curves studied by Cool, Draisma, Payne and Robeva in [CDPR12], and our results may be viewed as a generalized analogue of theirs, which simultaneously lends a geometric interpretation to their purely numerical ‘genericity’ condition. See Remark 5.4 for details.

In contrast, in the next section we will produce smaller families which we can show to be Brill-Noether general for arbitrary chain structures, under a generality hypothesis instead on the enriched structures.

We begin with a background proposition which holds for arbitrary curves of pseudocompact type.

Proposition 5.1. *The $(w_v)_v$ of Situation 2.6 can be chosen so that each w_v is nonnegative in every index.*

Proof. Let v_1 be any vertex of $\bar{\Gamma}$, and write $V(\bar{\Gamma}) = \{v_1, v_2, \dots, v_m\}$ with the ordering compatible with the distance from v_1 in $\bar{\Gamma}$. If w'_{v_1} is any admissible multidegree concentrated at v_1 , we can modify it into a w_{v_1} which is nonnegative in all indices other than v_1 as follows: for each $i > 1$, let $e_i \in E(\bar{\Gamma})$ be the edge adjacent to v_i in the direction of v_1 , and v'_i the other vertex adjacent to e_i . If w'_{v_1} is not nonnegative in index v_m , twist w'_{v_1} at (e_m, v'_m) the minimal number of times to make it nonnegative in index v_m . The minimality implies that the new w'_{v_1} is still concentrated at v_1 . Then repeat the process with v_{m-1} , v_{m-2} , and so forth, and ultimately we will arrive at the desired w_{v_1} . We then obtain w_{v_2} from w_{v_1} by twisting at (e_2, v_1) the maximal number of times possible without making it negative in index v_1 . We continue in this manner, inductively obtaining w_{v_i} from $w_{v'_i}$ by twisting at (e_i, v'_i) the maximal number of times possible without making it negative in index v'_i (note that we always have $v'_i = v_j$ for some $j < i$). \square

Our expected dimension result for restricted chain structures is then the following.

Corollary 5.2. *In Situation 2.10, suppose further that*

- (I) *there are at most three edges of Γ connecting any given pair of vertices;*
- (II) *for any adjacent vertices v, v' of Γ , if v, v' are connected by edges $(e_i)_i$, then for any integers $(x_i)_i$ with $\sum_i x_i \mathbf{n}(e_i) = 0$, if there is a unique j with $x_j > 0$, then we have*

$$\sum_i [x_j \mathbf{n}(e_j) / \mathbf{n}(e_i)] > d;$$

- (III) *each (multi)marked component of X is strongly Brill-Noether general.*

Then the space of limit linear series on (X, \mathbf{n}) of degree d is pure of the expected dimension ρ .

Remark 5.3. Note that condition (II) above is vacuous when v, v' are connected by a single edge. When v, v' are connected by a pair of edges e_1, e_2 , then condition (II) amounts to requiring that

$$\frac{\text{lcm}(\mathbf{n}(e_1), \mathbf{n}(e_2))}{\mathbf{n}(e_1)} + \frac{\text{lcm}(\mathbf{n}(e_1), \mathbf{n}(e_2))}{\mathbf{n}(e_2)} > d.$$

See Remark 5.4 below for the relationship to [CDPR12].

Note also that even when v, v' are connected by three edges, it is easy to find values of the $\mathbf{n}(e_i)$ for which (II) is satisfied: for instance, one can take $1, n, n^2$ for any $n \geq d$, or $1, n, n'$ for n, n' relatively prime and at least equal to d .

Recall also that Theorem 3.3 (III)-(VI) gives explicit cases in which condition (III) above is known to be satisfied.

Proof. According to Theorem 4.1, condition (III) implies that it is enough to show that the gluing conditions impose the maximum codimension. We begin by describing what happens for a pair of nodes connected by two edges. Condition (I) implies that each nontrivial gluing condition is expected to impose codimension 1; furthermore, such a condition can be imposed only when

$$\deg D_{i+1}^{(e,v)} - D_i^{(e,v)} = 2$$

for some i , where v is as in (II) and e is the edge connecting v to v' in $\bar{\Gamma}$. Now, we claim that we get at most one such nontrivial condition for any pair of components.

Indeed, this is a consequence of condition (II) and Proposition 5.1, since the way the $D_{\bullet}^{(e,v)}$ is defined, in order to have

$$\deg D_{i+1}^{(e,v)} - D_i^{(e,v)} = 2,$$

we need to have $\mathbf{n}(e_1)|(\sigma(e_1, v)\mu_v(e_1) + i)$ and $\mathbf{n}(e_2)|(\sigma(e_2, v)\mu_v(e_2) + i)$; if this occurs for $i_1 > i_2$, then $\text{lcm}(\mathbf{n}(e_1), \mathbf{n}(e_2))|(i_1 - i_2)$. But in order for a given $\deg D_i^{(e,v)}$ to occur in the multivanishing sequence, we must have $\deg D_i^{(e,v)} \leq d_v \leq d$, where d_v is the value of w_v in index v . Now,

$$\begin{aligned} \deg D_{i_1}^{(e,v)} - \deg D_{i_2}^{(e,v)} &= \frac{i_1 - i_2}{\mathbf{n}(e_1)} + \frac{i_1 - i_2}{\mathbf{n}(e_2)} \\ &\geq \text{lcm}(\mathbf{n}(e_1), \mathbf{n}(e_2)) (1/\mathbf{n}(e_1) + 1/\mathbf{n}(e_2)), \end{aligned}$$

so (III) implies that this cannot occur, proving the claim.

Now, suppose v and v' are connected by edges e_1 and e_2 , and we are given linear series with the appropriate multivanishing on Z_v and $Z_{v'}$. By the claim above, we have at most a single gluing condition at $Z_v \cap Z_{v'}$, and there are two cases to consider: if the sections determining the directions of gluing on Z_v and $Z_{v'}$ are nonvanishing at both P_{e_1} and P_{e_2} , then the gluing condition imposes the desired codimension 1 on the choice of global line bundle gluing at $Z_v \cap Z_{v'}$. Otherwise, the given gluing condition can be considered degenerate, since it is either impossible to glue, or every gluing of line bundles satisfies the required gluing condition. We thus stratify the space of tuples of linear series on the Z_v having the required multivanishing sequences based on which gluing conditions are degenerate. The nondegenerate cases all visibly independently impose codimension 1 on the choice of gluing for the global line bundle, while we see that for every degenerate gluing condition our stratum itself has its dimension reduced by (at least) 1, again independently. Indeed, the condition of having an extra order of vanishing at P_{e_1} or P_{e_2} can be expressed with a modified multivanishing sequence by the insertion of an addition divisor; the expected codimension of the modified multivanishing sequence is 1 greater than before, and by our strong Brill-Noether generality hypothesis, this additional codimension is in fact realized, as desired.

Now we consider what happens when some v, v' is connected by three edges e_1, e_2, e_3 . The argument is similar to the above, except that the analysis of gluing conditions is somewhat more complicated. Let e denote the edge connecting v and v' in $\bar{\Gamma}$; a nontrivial gluing condition may impose codimension 1 or codimension 2, depending partly on whether

$$D_{i+1}^{(e,v)} - D_i^{(e,v)}$$

has degree 2 or 3. Now, we claim that the inequalities of condition (II) imply that the latter case can occur at most once, and the former case can occur at most twice. Moreover, if the latter case occurs, then the former case does not, and if the former case does occur twice, the corresponding (reduced) degree-2 divisors cannot be supported on the same pair of points. Indeed, most of these assertions follow from the case that one of the x_j is zero by the same argument as above, which implies that there cannot be any pair of indices i such that $D_{i+1}^{(e,v)} - D_i^{(e,v)}$ contains a given pair of points P_{e_j} and $P_{e_{j'}}$. The only difference is that with three nodes,

the relevant formula for the change in divisor degrees is

$$\deg D_{i_1}^{(e,v)} - \deg D_{i_2}^{(e,v)} \geq \sum_{j=1}^3 \left\lfloor \frac{i_1 - i_2}{n(e_j)} \right\rfloor.$$

The only remaining possibility to consider is that $D_{i+1}^{(e,v)} - D_i^{(e,v)}$ has degree 2 three times, supported at all three possible pairs of points. But this is ruled out by condition (II) in the case that all x_j are nonzero.

Now, we proceed as before. Degeneracy can be more complicated if we have

$$\deg D_{i+1}^{(e,v)} - D_i^{(e,v)} = 3 :$$

if the relevant multivanishing index occurs once, the corresponding one-dimensional space might vanish at one or two of the points in $D_{i+1}^{(e,v)} - D_i^{(e,v)}$; if the relevant multivanishing index occurs twice, the corresponding two-dimensional space might vanish at one of the points, or it might contain a one-dimensional subspace vanishing at two of the points. All of these cases can be treated via the strong Brill-Noether generality hypothesis as before, and under hypothesis (IV) we see that they each impose the expected codimension, which is respectively 1, 2, 2 and 1. Gluing conditions can also be more complicated: depending on the number of repetitions in complementary terms of the multivanishing sequences on Z_v and $Z_{v'}$, we could obtain nontrivial gluing conditions in terms of agreement of a pair of points or a pair of lines in the projective plane, or in terms of a point lying on a line; the first two conditions have expected codimension 2, while the latter has expected codimension 1. However, in all combinations of degeneracy type and gluing conditions, we see that either a given gluing becomes impossible, or the degeneracy on both Z_v and $Z_{v'}$ imposes enough codimension to make up for lost gluing conditions. Thus, we obtain the desired gluing codimension in the case that

$$\deg D_{i+1}^{(e,v)} - D_i^{(e,v)} = 3$$

for some i .

The only other new case is that $D_{i+1}^{(e,v)} - D_i^{(e,v)}$ has degree 2 twice for a given v, v' . Here again the expected gluing codimension is 2, and if there is any degeneracy, the same analysis as before gives the desired codimension. On the other hand, in the nondegenerate case, one easily sees that the pair of nontrivial gluing conditions (supported on two distinct pair of points) uniquely determines the gluing of the line bundles, and hence imposes codimension 2, as desired. \square

Remark 5.4. If we set $d = 2g - 2$ in condition (II) of Corollary 5.2, then in light of Remark 5.3 and the discussion at the end of §2, this recovers precisely the ‘genericity’ condition of [CDPR12]. Thus, our proof of Corollary 5.2 is in essence showing that for the curves they consider, and under their genericity conditions, gluing conditions always automatically impose the expected codimension, and do so purely on the level of line bundle gluings. It seems reasonable to expect that this sort of situation should be highly predictive for when metric graphs will be Brill-Noether general.

We conclude this section with an explicit criterion for Brill-Noether generality in terms of degenerations, simultaneously generalizing previous conditions due to Eisenbud and Harris, Welters [Wel85], and Cools, Draisma, Payne and Robeva. For

simplicity, we restrict to characteristic 0, although this only comes into condition (II), and the positive-characteristic analogue follows identically from Theorem 3.3.

Corollary 5.5. *Let $\pi : X \rightarrow B$ be a flat, proper morphism with B the spectrum of a discrete valuation ring over a field of characteristic 0, generic fiber X_η a smooth curve, and special fiber X_0 a nodal curve.*

Suppose that each component of X_0 has one of the following forms:

- (I) *a rational curve meeting at most two other components of X_0 in at most three nodes each;*
- (II) *a rational curve meeting every other component of X_0 in at most one node;*
- (III) *an elliptic curve meeting at most one other component of X_0 in a single node, or meeting two other components of X_0 in single nodes which do not differ by m -torsion for any $m \leq 2g - 2$;*
- (IV) *a genus-2 curve meeting one other component of X_0 at a single point which is not a Weierstrass point.*

Suppose further that the chain structure \mathbf{n} induced on X_0 by the singularities of X satisfies the inequalities of condition (II) of Corollary 5.2, with $d = 2g - 2$.

Then X_η is Brill-Noether general.

Proof. According to Theorem 3.3 (III)-(VI), the hypotheses on the components of X_0 imply that they are strongly Brill-Noether general, so condition (III) of Corollary 5.2 is satisfied. Condition (I) is visibly satisfied, and we finally observe that condition (II) is invariant under scaling of the chain structure, so Corollary 5.2 tells us that we have the expected dimension for all limit linear series spaces for $(X_0, c\mathbf{n})$, with c any positive integer. It then follows from the specialization result Corollary 3.14 of [Oss14b] that X_η is Brill-Noether general. \square

6. CURVES WITH FEW NODES: GENERAL ENRICHED STRUCTURES

Continuing with the study of curves with few nodes between given pairs of components, we consider a narrower family of curves, and show that every curve in the family is Brill-Noether general if equipped with a general enriched structure, irrespective of chain structure.

We begin with a straightforward proposition on the structure of choices of enriched structures, which holds for arbitrary curves of pseudocompact type.

Proposition 6.1. *In Situation 2.10, the space of enriched structures on (X, \mathbf{n}) is canonically identified with the product of the spaces of enriched structures on subcurves $Z_{v,v'} \subseteq \tilde{X}$, where v, v' are adjacent vertices of Γ , and $Z_{v,v'}$ is the subcurve of \tilde{X} containing Z_v and $Z_{v'}$, and all chains of rational components connecting them.*

If $v, v' \in V(\Gamma)$ are connected by m edges e_1, \dots, e_m , then $(k^)^{m-1}$ acts freely on the space of enriched structures on $Z_{v,v'}$ as follows: given $(\lambda_1, \dots, \lambda_{m-1}) \in (k^*)^{m-1}$, we scale the gluing map defining \mathcal{O}_v at P_{e_i} by λ_i and the gluing map defining $\mathcal{O}_{v'}$ at P_{e_i} by λ_i^{-1} for each $i \leq m - 1$.*

Our expected dimension result is then the following.

Corollary 6.2. *In Situation 2.10, suppose further that*

- (I) *there are at most two edges of Γ connecting any given pair of vertices;*
- (II) *for every $v \in V(\Gamma)$, if there is some $v' \in V(\Gamma)$ which is connected to v by two edges, then Z_v is rational and v has valence at most three in Γ ;*

- (III) the enriched structure on (X, \mathbf{n}) is general;
- (IV) each (multi)marked component of X is strongly Brill-Noether general.

Then the space of limit linear series on (X, \mathbf{n}) of degree d is pure of the expected dimension ρ .

Thus, the curves considered in Corollary 6.2 are almost of compact type, except that an elliptic component containing at most two nodes may be replaced by a pair of rational components meeting each other at a pair of nodes. Nonetheless, this contains (noncompact-type) curves of every genus, some of which have arisen in other contexts; see Example 6.4.

The following lemma describes the behavior of gluing conditions in the case of interest, saying in essence that with fixed (multi)vanishing sequences, we will always have some gluing directions fixed, and the rest varying freely.

Lemma 6.3. *Let X be rational, with distinct points P_1, P_2 and Q . Fix also divisor sequences D_\bullet^1, D_\bullet^2 as in Definition 1.1 and supported on the P_i and Q respectively, $r, d > 0$, and multivanishing sequences a^1 and a^2 , and let $G_d^{r,\circ}(X, (D_\bullet^i, a^i)_i)$ denote the open subscheme of $G_d^r(X, (D_\bullet^i, a^i)_i)$ consisting of linear series with multivanishing sequences along the D_\bullet^i given precisely by a^i . Let S be the set of indices j such that $\deg D_{j+1}^1 - \deg D_j^1 = 2$, and $\deg D_j^1$ occurs exactly once in a^1 . Then there exists a subset $S' \subseteq S$ such that the image of the natural map*

$$(6.1) \quad G_d^{r,\circ}(X, (D_\bullet^i, a^i)_i) \rightarrow \prod_{j \in S} \mathbb{P}(\mathcal{O}_X(d)(-D_j^1)/\mathcal{O}_X(d)(-D_{j+1}^1))$$

is a fiber of the projection to

$$\prod_{j \in S'} \mathbb{P}(\mathcal{O}_X(d)(-D_j^1)/\mathcal{O}_X(d)(-D_{j+1}^1)).$$

Moreover, the image in

$$\prod_{j \in S'} \mathbb{P}(\mathcal{O}_X(d)(-D_j^1)/\mathcal{O}_X(d)(-D_{j+1}^1))$$

is nondegenerate in the sense that it is a tuple of subspaces, each of which has no more vanishing at either P_1 or P_2 than that prescribed by a^1 .

Finally, the induced map

$$(6.2) \quad G_d^{r,\circ}(X, (D_\bullet^i, a^i)_i) \rightarrow \prod_{j \in S \setminus S'} \mathbb{P}(\mathcal{O}_X(d)(-D_j^1)/\mathcal{O}_X(d)(-D_{j+1}^1))$$

has every fiber of codimension equal to $\#(S \setminus S')$.

Proof. First observe that the space $G_d^r(X, (D_\bullet^i, a^i)_i)$, as the intersection of two Schubert varieties corresponding to complementary flags, is a Richardson variety and hence irreducible. We refer to §§2.2 and 2.5 (particularly Lemma 2.6) of Vakil [Vak06] for background on intersections of pairs of Schubert varieties, expressed combinatorially in terms of positions of black and white checkers. By irreducibility, it is enough to prove the first statement of the lemma for any open subset of $G_d^{r,\circ}(X, (D_\bullet^i, a^i)_i)$. Let $G_d^{r,\circ\circ}(X, (D_\bullet^i, a^i)_i)$ be the largest cell, consisting of spaces V admitting a basis e_0, \dots, e_r such that e_i vanishes to order exactly a_i^1 along D_\bullet^1 , and to order exactly a_{r-i}^2 along D_\bullet^2 . Now, in order for $G_d^r(X, (D_\bullet^i, a^i)_i)$ to be nonempty we must have $a_i^1 + a_{r-i}^2 \leq d$ for all i ; let S' be the set of $j \in S$ such that we have

$\deg D_j^1 = a_i^1$ with $a_i^1 + a_{r-i}^2 = d$, and let T be the set of i such that $a_i^1 = \deg D_j^1$ for some $j \in S'$.

If $i \in T$, the choice of e_i is unique up to scalar, and since a_i^1 is nonrepeated, the image of (6.1) in index a_i^1 is thus unique on $G_d^{r,\circ\circ}(X, (D_\bullet^i, a^i)_i)$. Moreover, e_i cannot have any additional vanishing at P_1 or P_2 . On the other hand, if $i \notin T$, then $h^0(X, \mathcal{O}_X(d)(-D_j^1)) \geq 2$, and therefore the choices of e_i surject onto the nonzero elements of $\mathcal{O}_X(d)(-D_j^1)/\mathcal{O}_X(d)(-D_{j+1}^1)$. Because the choices of e_i are independent for different i , we conclude that the image of $G_d^{r,\circ\circ}(X, (D_\bullet^i, a^i)_i)$ under (6.1) is a fiber of the projection to $\prod_{j \in S'} \mathbb{P}(\mathcal{O}_X(d)(-D_j^1)/\mathcal{O}_X(d)(-D_{j+1}^1))$, as desired, and has the claimed nondegeneracy property.

The proof of the assertion on the fibers (6.2) is similar, but a bit more involved: if we fix a point

$$z \in \prod_{j \notin S'} \mathbb{P}(\mathcal{O}_X(d)(-D_j^1)/\mathcal{O}_X(d)(-D_{j+1}^1)),$$

then the fiber over z still corresponds to (an open subset of) an intersection of two Schubert varieties, with one still corresponding to the flag determined by D_\bullet^2 , and the other corresponding to a refinement of the flag determined by D_\bullet^1 , with exactly $\#\{S \setminus S'\}$ (non-repeated) entries of a^1 increased by 1. Let $(a^1)'$ denote this modified multivanishing sequence. The new flags need not be transverse, and the intersection may no longer be irreducible, but we can study its maximal cells as described in [Vak06], and it suffices to see that their dimensions are still as expected (i.e., are the same as in the case that the flags are transverse). Now, in Vakil's notation the non-transversality of the refined flag is expressed in terms of black checkers, by starting with the default position along the antidiagonal (corresponding to two transverse flags), and allowing some (disjoint) pairs of adjacent checkers to have their x and y -coordinates swapped. Possible white checkers positions correspond to permutations σ of $\{0, \dots, r\}$, with the $(i+1)$ st white checker in position $((a^1)')_{\sigma(i)} + 1, a_{r-i}^2 + 1$. We also have the additional constraint that each white checker must have a black checker weakly above it, and a black checker weakly to the left of it. We first consider the case that $\sigma = \text{id}$, which corresponds to the maximal cell in the case of transverse flags. We see that because z only specified directions away from S' , the entries of $(a^1)'$ which were increased by 1 did not correspond to white checkers which were already on the antidiagonal, so the new configuration of white checkers lies entirely on or below the antidiagonal. In this case, we see that our adjacent swaps of black checkers does not affect the number of black checkers dominated by any given white one, so the dimension of this cell is as expected. Finally, given any σ , if we write it as a product of disjoint cycles we see that the dimension of the corresponding cell can be analyzed cycle by cycle, and any increase in black checkers dominating by the white ones is always at least cancelled by a corresponding increase in domination of other white checkers, so the dimension cannot increase. We thus conclude that we have the expected codimension, as desired. \square

Proof of Corollary 6.2. According to Theorem 4.1, it is enough to verify that if we stratify the space of limit linear series by multivanishing sequences, the gluing conditions impose the maximal possible codimension. Note that because there are finitely many strata, it is enough to prove that a general enriched structure has the desired behavior one stratum at a time. According to Proposition 6.1, we can describe enriched structures in terms of one pair of adjacent components at a time.

Now, if v and v' are connected by 2 edges, then modifying the enriched structure by scaling as in the proposition will change the resulting gluing maps by successive powers of the given scalar. Thus, for any two fixed gluing directions on Z_v and on $Z_{v'}$, a general enriched structure will not allow both conditions to be simultaneously satisfied. According to Lemma 6.3, with the multivansishing sequences fixed on both Z_v and $Z_{v'}$, we find a finite set of fixed gluing directions on each of Z_v and $Z_{v'}$, and the generality condition on the enriched structure simply requires that if two or more such fixed directions on Z_v are paired with those on $Z_{v'}$, that there not exist any line bundle gluing simultaneously satisfying the gluing conditions.

With this generality condition, we have ensured that if a given stratum is nonempty, each pair $Z_v, Z_{v'}$ as above has at most one gluing condition which comes from fixed directions on both sides. Such a condition, if it occurs, uniquely determines the line bundle gluing between Z_v and $Z_{v'}$, while any gluing condition which is not fixed on one side or the other imposes the desired codimension, independently of one another, by the last part of Lemma 6.3. We thus conclude that the gluing conditions impose the maximal codimension, as desired. \square

Example 6.4. The prototypical example of the curves treated by Corollary 6.2 is a chain of rational curves in which the components alternate meeting in two nodes or one node. Such curves were considered for instance in Jensen and Payne's tropical approach to the Gieseker-Petri theorem [JP]. Corollary 5.2 and 6.2 imply that such curves are Brill-Noether general with suitable hypotheses on either the chain or enriched structure. However, we see that if arbitrary chain and enriched structures are allowed, they are not Brill-Noether general. Indeed, such curves are in the closure of the hyperelliptic locus (for instance, one easily checks that they carry degree-2 admissible covers of a chain of projective lines), so general theory tells us that there must exist limit \mathfrak{g}_2^1 s on them.

However, it is just as easy to see this explicitly: take the trivial chain structure (meaning $\mathbf{n}(e) = 1$ for all e), and choose concentrated multidegrees of 2 on the main component and 0 on every other component. Take (multi)vanishing sequence 0, 2 at each node or pair of nodes. On each interior component, we are looking for 2-dimensional spaces of polynomials of degree 2 which have an element vanishing to order 2 at one point, and another element vanishing simultaneously at two other points. Such spaces are of course uniquely determined. At the ends, we have only the condition that our space contain an element vanishing simultaneously at two points, so have a projective line of choices. Now, there are no gluing conditions at the pairs of components connected by a single node, but at the pairs connected by two nodes, we have two gluing conditions, with one coming from each term of the multivansishing sequence. The first gluing condition uniquely determined a choice of gluing for the line bundle, but the second gluing condition will only be satisfied for special enriched structures, except on the first and last components. On the interior components, one checks easily that there does always exist enriched structures for which the two gluing conditions are compatible, so we do in fact get a limit \mathfrak{g}_2^1 , as claimed.

7. BINARY CURVES

As the opposite extreme from the case where there are few nodes connecting any pair of components, we now consider the case of binary curves:

Situation 7.1. Suppose that X is obtained by gluing rational curves Z_1 and Z_2 to one another at $g + 1$ nodes.

In Situation 7.1, a $\mathfrak{g}_{(d_1, d_2)}^r$ refers to a line bundle on X of degree d_i on each Z_i , together with an $(r + 1)$ -dimensional space of global sections. In the case of binary curves, in contrast to the previous sections, it turns out that there is only a single gluing condition which can fail to impose maximal codimension, and this condition is insensitive to chain structures or enriched structures. We will show that for any binary curve, many components of the spaces of limit linear series have the expected dimension. The following definition turns out to determine the behavior of the remaining components.

Definition 7.2. In Situation 7.1, we say that X is **weakly Brill-Noether general** for a given $r > 0$ if for all (d_1, d_2) with

$$0 \leq d_i \leq g - 1, \quad \text{for } i = 1, 2$$

the space of $\mathfrak{g}_{(d_1, d_2)}^r$ s on X has the expected dimension ρ .

The reason for our terminology is that the range of multidegrees we consider is smaller than would be required in order to obtain a proof of the Brill-Noether theorem considering only naive linear series, as is done by Caporaso in [Cap10]. However, our main result on binary curves is that using our theory of limit linear series, the above multidegrees are sufficient.

Corollary 7.3. *In Situation 7.1, any component of a limit linear series space on X for which the general member has nonconstant multivanishing sequences has the expected dimension ρ .*

Moreover, if for a given r we have that X and all its partial normalizations are weakly Brill-Noether general, then for the same r , all spaces of limit linear series on X have the expected dimension ρ .

Proof. According to Theorem 4.1, it is enough to fix all discrete invariants, including choices of multivanishing sequences, and to show that the gluing conditions impose the maximal codimension. We simplify notation as follows: for $i = 1, 2$ let d_i denote the d_v of the proof of Theorem 4.1 applied to the components Z_i , and similarly let D_i^\bullet be the sequences of effective divisors on Z_i , indexed from 0 to $b + 1$. Set $\mathcal{L}^i := \mathcal{O}_{Z_i}(d_i)$, and let a^i be the multivanishing sequence on Z_i for $i = 1, 2$. Finally, let the sequences r^i be obtained from the a^i as in Definition 3.2. It is clearly enough to show that the gluing condition (II) of Definition 2.16 imposes the correct codimension for each choice of line bundle gluing, so we fix such a gluing.

Let G be the product over all critical j of

$$G(r_j^1, \mathcal{L}^1(-D_j^1)/\mathcal{L}^1(-D_{j+1}^1)) \times G(r_{b-j}^2, \mathcal{L}^2(-D_{b-j}^2)/\mathcal{L}^2(-D_{b+1-j}^2)),$$

so that our gluing conditions can be expressed as (the preimage of) a subvariety of G . Now, because the Z_i are rational, for any j with $\deg D_{j+1}^1 \leq d_1 + 1$ we have

$$H^0(Z_1, \mathcal{L}^1(-D_j^1))/H^0(Z_1, \mathcal{L}^1(-D_{j+1}^1)) \xrightarrow{\sim} \mathcal{L}^1(-D_j^1)/\mathcal{L}^1(-D_{j+1}^1).$$

Let j_1 be maximal with $\deg D_j^1 \leq d_1 + 1$. Then if $j = j_1$, we have that the space $H^0(Z_1, \mathcal{L}^1(-D_j^1))/H^0(Z_1, \mathcal{L}^1(-D_{j+1}^1))$ imbeds into $\mathcal{L}^1(-D_j^1)/\mathcal{L}^1(-D_{j+1}^1)$ as a proper linear subspace. Of course, if $j > j_1$, then $H^0(Z_1, \mathcal{L}^1(-D_j^1)) = 0$, so we see that in fact the space of choices of V^1 can be expressed as the product over

critical j of $G(r_j^1, W_j^1)$, where $W_j^1 \subseteq \mathcal{L}^1(-D_j^1)/\mathcal{L}^1(-D_{j+1}^1)$ is a subspace which is the entire space for $j < j_1$, and zero for $j > j_1$. Letting j_2 be maximal such that $\deg D_j^2 \leq d_2 + 1$, we have an analogous statement for Z_2 , with subspaces $W_j^2 \subseteq \mathcal{L}^2(-D_j^2)/\mathcal{L}^2(-D_{j+1}^2)$. Thus, if P is the space of choices of (V^1, V^2) with the required multivanishing sequences, we see that P is in fact a subvariety of G which is itself a product. It is thus enough to work one index at a time, verifying that the image of P in the relevant index meets the gluing condition subvariety with the correct codimension.

Now, if we have j with $j < j_1$ and $b - j < j_2$, the image of P is all of $G(r_j^1, \mathcal{L}^1(-D_j^1)/\mathcal{L}^1(-D_{j+1}^1)) \times G(r_{b-j}^2, \mathcal{L}^2(-D_{b-j}^2)/\mathcal{L}^2(-D_{b+1-j}^2))$ (and this is independent of any conditions on other values of j), so there is certainly no problem with gluing codimension for such values of j . If $j = j_1$ and $b - j < j_2$, the image is $G(r_{j_1}^1, W_{j_1}^1) \times G(r_{b-j_1}^2, \mathcal{L}^2(-D_{b-j_1}^2)/\mathcal{L}^2(-D_{b+1-j_1}^2))$, and it is straightforward to check that the gluing codimension is still as desired. The same still holds if $j < j_1$ and $b - j = j_2$. Values of j with $j > j_1$ or $b - j > j_2$ cannot occur in the multivanishing sequence, so the only case left to consider is that $j = j_1$ and $b - j = j_2$. The expected codimension of gluing in this case is only positive if $\deg D_{j_1}^1$ occurs in a^1 (in which case it is necessarily a_r^1), and if $\deg D_{b-j_1}^2$ occurs in a^2 (in which case it is a_r^2).

However, we see that this can only happen under very restrictive circumstances: indeed, if $b - j_1 = j_2$, then for $j > j_1$ we cannot have $\deg D_j^1$ occurring in a^1 , while for $j < j_1$ we have $b - j > j_2$, so $\deg D_{b-j}^2$ cannot occur in a^2 . On the other hand, (2.1) implies that if $a_r^1 = \deg D_{j_1}^1$, then $a_0^2 \geq \deg D_{b-j_1}^2$, so we find that $a_\ell^2 = \deg D_{b-j_1}^2$ for all ℓ , and similarly $a_\ell^1 = \deg D_{j_1}^1$ for all ℓ . But in this case, our space of limit linear series (if we now let the gluing of the line bundle vary again) is almost the same as the space of naive linear series on X' , of multidegree $(d_1 - \deg D_{j_1}^1, d_2 - \deg D_{b-j_1}^2)$, where X' is the partial normalization of X at the nodes not occurring in the support of $\deg D_{j_1+1}^1 - \deg D_{j_1}^1$. Indeed, the only difference is that the latter space does not remember the choice of line bundle gluing at the normalized nodes, so our limit linear series space is smooth over the latter space, and the expected codimension of gluing is the same in both. Now, because $\deg D_{j_1+1}^1 > d_1 + 1$, and $\deg D_{j_1+1}^1 - \deg D_{j_1}^1 \leq g + 1$, we have that $d_1 - \deg D_{j_1}^1 \leq g - 1$, and similarly $d_2 - \deg D_{b-j_1}^2 \leq g - 1$, so we conclude the desired reduction statement. \square

Putting together Corollary 7.3 with a theorem of Caporaso, we find that we have good behavior for $r = 1, 2$, as follows.

Corollary 7.4. *In the situation of Corollary 7.3, suppose that X is a general binary curve, and $r \leq 2$. Then all moduli spaces of limit linear series on X have the expected dimension ρ .*

Proof. According to Theorem 27 of [Cap10], when X is general, usual linear series of multidegrees (d_1, d_2) on X have the expected dimension whenever $r \leq 2$ and

$$\frac{d - g - 1}{2} \leq d_i \leq \frac{d + g + 1}{2}$$

for $i = 1, 2$. One checks easily that these inequalities are weaker than $0 \leq d_i \leq g - 1$ for $i = 1, 2$. The asserted result thus follows from Corollary 7.3. \square

Remark 7.5. The complete independence of Corollary 7.4 from both enriched structures and chain structures may be a bit surprising, insofar as it differs both from the behavior of the families considered in §§5 and 6 and from the intuition that even a curve in the closure of a Brill-Noether divisor could behave as a general curve if approached from a suitably general direction.

However, we observe that this behavior is in fact somewhat predictable, insofar as our theory is a direct generalization of the Eisenbud-Harris theory for curves of compact type, and even of usual linear series for smooth curves. Indeed, in both cases neither enriched structures nor chain structures are relevant, and yet non-general curves need not be Brill-Noether general.

Example 7.6. While we hope that the condition $r \leq 2$ in Corollary 7.4 may be removed via further analysis, the generality condition on X is certainly necessary. Indeed, if X is obtained by gluing together two copies of the same marked rational curve, we see that X always has a \mathfrak{g}_2^1 of multidegree $(1, 1)$, which may be considered as a limit linear series with $b = 0$. As long as $g > 2$, we have $\rho < 0$, so this violates Corollary 7.4. Note that this happens regardless of enriched structure or chain structure, so again underlines the peculiar irrelevance of enriched structures in the case of binary curves, discussed in Remark 7.5.

Remark 7.7. While the way we have presented Corollary 7.4 requires an imprecise generality hypothesis, in fact one can analyze the situation quite explicitly, since the weak Brill-Noether generality hypothesis amounts to studying whether we can find $(r + 1)$ -dimensional spaces of polynomials of degree d_1 and d_2 which can be made to glue to one another at the chosen points, under suitable choices of gluings. This can be set up as an explicit linear algebra problem, leading to the potential for more precise criteria for Brill-Noether generality, especially in fixed r . For instance, one can easily check via this approach that the existence of a $g_{(1,1)}^1$ (the only possibility for a g_2^1) is equivalent to the marked points on Z_1 and Z_2 being the same, up to automorphism.

Remark 7.8. Note that despite the inequalities in the definition of being weakly Brill-Noether general, it does not follow from Corollary 7.3 that all spaces of limit linear series on binary curves have dimension ρ when $d > 2g - 2$. The reason is that the reduction process in the proof of Corollary 7.3 may decrease the degree, or put differently, one can imbed spaces of lower-degree linear series into spaces of higher-degree limit linear series by imposing extra multivanishing conditions. This is not new to our notion of limit linear series – in fact, exactly the same phenomenon occurs for Eisenbud-Harris limit linear series.

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